Miura type transformations and homogeneous spaces

Sergey A Igonin

Department of Mathematics, Utrecht University, PO Box 80010, 3508 TA Utrecht,
The Netherlands

E-mail: igonin@mccme.ru

Abstract
We relate Miura type transformations (MTs) over an evolution system to its
zero-curvature representations with values in Lie algebras $\mathfrak{g}$. We prove that
certain homogeneous spaces of $\mathfrak{g}$ produce MTs and show how to distinguish
these spaces. For a scalar translation-invariant evolution equation, this allows
us to classify all MTs in terms of homogeneous spaces of the Wahlquist–
Estabrook algebra of the equation. For other evolution systems this allows us
to construct some MTs. As an example, we study MTs over the KdV equation,
a fifth-order equation of Harry Dym type, and the coupled KdV–mKdV system
of Kersten and Krasilshchik.

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1. Introduction

In this paper, we study $(1 + 1)$-dimensional evolution systems

$$\frac{\partial v^i}{\partial t} = R^i\left(x, t, v^1, \ldots, v^k, v_1^1, \ldots, v_1^k, \ldots, v_r^1, \ldots, v_r^k\right), \tag{1}$$

$$\frac{\partial u^i}{\partial t} = P^i\left(x, t, u^1, \ldots, u^k, u_1^1, \ldots, u_1^k, \ldots, u_p^1, \ldots, u_p^k\right),$$

$$v_j^i = \frac{\partial^j v^i}{\partial x^j}, \qquad u_j^i = \frac{\partial^j u^i}{\partial x^j}, \qquad i = 1, \ldots, k, \tag{2}$$

and transformations

$$u^i = S^i\left(x, t, v^1, \ldots, v^k, v_1^1, \ldots, v_1^k, \ldots, v_n^1, \ldots, v_n^k\right) \tag{3}$$

such that if $v^1, \ldots, v^k$ satisfy (1) then (3) satisfy (2). By analogy with the classical Miura
transformation connecting the KdV and mKdV equations, such transformations are called
Miura type transformations (MTs in short).

It is well known that MTs play an important role in the theory of integrable evolution systems. Chains of MTs generate Bäcklund transformations [13, 14].

If P^i, R^i, S^i do not depend on x and t, the MT is called *translation invariant*.

In the scalar case $k = 1$, the problem to find all pairs (2), (3) for a given system (1) was solved efficiently in [16].

We study the opposite problem: for a given system (2), how do we find all pairs (1), (3)? It seems that this problem was systematically studied only for translation-invariant MTs of the linear equation $u_t = u_3$ [9] and the KdV equation [2, 3, 9]. In particular, a set of MTs over the KdV equation was constructed using homogeneous spaces of certain Lie groups [2] and loop groups [3]. These MTs do not exhaust all translation-invariant MTs over the KdV equation, but, knowing all integrable equations of the form $u_t = u_3 + f(u, u_1, u_2)$, one can show that all other MTs can be obtained from these by the introduction of a potential.

We relate MTs (3) with zero-curvature representations of (2) (ZCRs in short) dependent on $x, t, u^i_j, j \leqslant p - 1$. It turns out that such a ZCR with values in a Lie algebra $\mathfrak{g}$ and a certain representation of $\mathfrak{g}$ by vector fields on a manifold W determine a MT. Among other requirements, the image of $\mathfrak{g}$ under the representation must span the tangent spaces of W, that is, the manifold W is a homogeneous space of $\mathfrak{g}$.

This construction is not surprising if one recalls the theory of coverings of PDEs [6, 12]. It is becoming clear [6] that each covering is determined by a $\mathfrak{g}$-valued ZCR and a vector field representation of some Lie algebra $\mathfrak{g}$. Since MTs are a particular type of coverings, it remains to determine which ZCRs and representations lead to MTs. However, in order to be self-contained, we do not introduce the coverings terminology and work in local coordinates.

If $k = 1$ and (2) is translation invariant, we prove that every MT arises in this way from the 'universal' ZCR with values in the Wahlquist–Estabrook algebra of (2). This allows us to reduce the classification of MTs to the classification of certain homogeneous spaces of the Wahlquist–Estabrook algebra. As an example, we obtain that any (not necessarily translation-invariant) MT over the KdV equation is of order not greater than 3. Also, we recover the Lie groups of [2] as the Lie groups associated with some quotients of the Wahlquist–Estabrook algebra and explain why the method of [2] does not give all translation-invariant MTs over the KdV equation.

Another considered example is the equation $u_t = u^{\frac{5}{2}} u_5$ [4, 11]. Using its Wahlquist–Estabrook algebra computed in [4], we show that any MT over this equation is of order not greater than 5 and construct a MT of order 3. The corresponding modified equation (39) may be a new integrable equation.

For non-scalar systems, we obtain only a sufficient condition for a ZCR to define a MT. As an example, we construct a MT over the coupled KdV–mKdV system of Kersten and Krasilshchik [8]. This MT arises from the ZCR obtained in [7]. Again, the corresponding modified system may be new.

There are also more general transformations of evolution equations, where one changes not only dependent variables, but also the x variable (see, e.g., [5, 16, 15]). It remains an interesting open question whether our Lie algebraic methods can be generalized for studying these transformations.

The paper is organized as follows. In section 2, we study Lie algebras actions of special type that will later turn out to be responsible for MTs. In sections 3 and 4 we describe MTs of scalar evolution equations and their relations with Wahlquist–Estabrook algebras. Finally, in section 5 we study MTs of non-scalar evolution systems.

2. Actions of Lie algebras on manifolds

Recall that an *action* of a Lie algebra $\mathfrak{g}$ on a manifold W is a homomorphism $\rho\colon \mathfrak{g} \to D(W)$ to the Lie algebra $D(W)$ of vector fields on W. The action is said to be *transitive* if for each point $a \in W$ the mapping

$$\mathrm{ev}_{\rho,a}\colon \mathfrak{g} \to T_a W, \qquad g \mapsto \rho(g)_a,$$

is surjective. Two actions $\rho_i\colon \mathfrak{g} \to D(W_i)$, $i = 1, 2$, are said to be *isomorphic* if there is a diffeomorphism $\varphi\colon W_1 \to W_2$ such that $\rho_2 = \varphi_* \rho_1$.

Below our considerations are always local. The results are valid in both categories of smooth and complex-analytic manifolds. Depending on the category considered, all functions are supposed to be smooth or complex-analytic.

In what follows, we often consider a (possibly infinite) chain of subalgebras

$$\mathfrak{g}^1 \subset \mathfrak{g}^2 \subset \mathfrak{g}^3 \subset \cdots \subset \mathfrak{g} \tag{4}$$

and an action $\rho\colon \mathfrak{g} \to D(W)$. It easy to see that there is a non-empty open subset $W_c \subset W$ such that

$$\forall a, a' \in W_c, \qquad \forall i \quad \dim \mathrm{ev}_{\rho,a}(\mathfrak{g}^i) = \dim \mathrm{ev}_{\rho,a'}(\mathfrak{g}^i),$$
$$\dim \mathrm{ev}_{\rho,a}(\mathfrak{g}) = \dim \mathrm{ev}_{\rho,a'}(\mathfrak{g}). \tag{5}$$

Moreover, if W is connected and ρ is analytic, one can choose W_c to be dense in W. Since we study locally non-degenerate points only, below we always assume $W = W_c$.

Denote $m_i = \dim \mathrm{ev}_{\rho,a}(\mathfrak{g}^i)$ for $a \in W_c$. Due to inclusions (4) we have

$$m_1 \leqslant m_2 \leqslant m_3 \leqslant \cdots. \tag{6}$$

Lemma 1. *In the above-described situation, suppose that there is $V \in D(W)$ such that for each i the Lie algebra generated by the subspace $\rho(\mathfrak{g}^i) + [\rho(\mathfrak{g}^i), V]$ coincides with $\rho(\mathfrak{g}^{i+1})$. Set $n = \dim W$. Suppose that*

$$m_1 \geqslant s, \qquad m_{n-s} < n, \qquad \exists k \quad m_k = n. \tag{7}$$

Then

$$m_i = s + i - 1, \qquad i = 1, \ldots, n - s + 1. \tag{8}$$

Moreover, for each point $z \in W$ on a neighbourhood of z there is a function w such that

$$d_z w \neq 0, \qquad \rho(\mathfrak{g}^{n-s})(w) = 0,$$

and this function is unique up to a change $w \mapsto g(w)$. Set

$$\tilde{w}^i = V^{i-1}(w), \qquad i = 1, \ldots, n - s + 1.$$

One can find functions $\tilde{w}^{n-s+2}, \ldots, \tilde{w}^n$ such that $\tilde{w}^1, \ldots, \tilde{w}^n$ form a system of coordinates on a neighbourhood of z.

Proof. For a set S of vector fields denote by $\langle S \rangle$ the submodule (over the algebra of functions) of vector fields generated by S. By condition (5) and the Frobenius theorem, there are coordinates $w^1, \ldots, w^n$ on a neighbourhood of z such that

$$\forall i \quad \langle \rho(\mathfrak{g}^i) \rangle = \left\langle \frac{\partial}{\partial w^1}, \ldots, \frac{\partial}{\partial w^{m_i}} \right\rangle. \tag{9}$$

Denote module (9) by M_i. Suppose that $m_k = m_{k+1}$ for some k. Then $[M_k, V] \subset M_k$ and, therefore, $m_p = m_k$ for all $p \geqslant k$. Combining this property with (7) and (6), we obtain (8).

Now we can take $w = w^n$. Using the equalities

$$M_i + \langle [M_i, V] \rangle = M_{i+1}, \qquad i = 1, \ldots, n - s,$$

by induction on i one proves

$$M_{n-s-i+1}(\tilde{w}^i) = 0, \qquad \frac{\partial \tilde{w}^i}{\partial w^{n-i+1}} \neq 0, \quad i = 1, \ldots, n - s + 1,$$

where

$$M_0 = \left\langle \frac{\partial}{\partial w^1}, \ldots, \frac{\partial}{\partial w^{s-1}} \right\rangle.$$

Therefore, if we set $\tilde{w}^{n-s+1+i} = w^i$ for $i = 1, \ldots, s - 1$, the functions $\tilde{w}^1, \ldots, \tilde{w}^n$ will be local coordinates on a neighbourhood of z. $\qquad\square$

3. MTs from ZCRs of scalar evolution equations

Consider two scalar evolution equations

$$u_t = P(x, t, u, u_1, \ldots, u_p), \qquad u_k = \frac{\partial^k u}{\partial x^k}, \tag{10}$$

$$v_t = R(x, t, v, v_1, \ldots, v_r), \qquad v_k = \frac{\partial^k v}{\partial x^k}, \tag{11}$$

connected by a MT

$$u = S(x, t, v, v_1, \ldots, v_n). \tag{12}$$

The maximal integer n such that (12) depends non-trivially on v_n is called the *order* of the MT. A MT obtained from this one by a substitution $v \mapsto g(v)$ is said to be *equivalent* to the initial MT.

Introduce new variables

$$w^i = \frac{\partial^{i-1} v}{\partial x^{i-1}}, \qquad i = 1, \ldots, n, \tag{13}$$

and rewrite systems (11), (12) as follows:

$$\begin{aligned}
\frac{\partial w^i}{\partial x} &= w^{i+1}, \qquad i = 1, \ldots, n - 1, \\
\frac{\partial w^n}{\partial x} &= a(w^1, \ldots, w^n, x, t, u), \\
\frac{\partial w^i}{\partial t} &= b^i(w^1, \ldots, w^n, x, t, u, \ldots, u_{p-1}), \qquad i = 1, \ldots, n,
\end{aligned} \tag{14}$$

where p is the order of (10). And, vice versa, it is easily seen that any consistent system of this form with the non-degeneracy condition

$$\exists w_0^1, \ldots, w_0^n, x_0, t_0, u_0 \qquad \frac{\partial a}{\partial u}(w_0^1, \ldots, w_0^n, x_0, t_0, u_0) \neq 0 \tag{15}$$

determines a MT of order n for (10) as follows:

- substitute (13) to (14),
- taking into account (15), from equation (14) express locally $u = S(x, t, v, v_1, \ldots, v_n)$,
- let $D = \sum_{i \geqslant 0} v_{i+1} \partial / \partial v_i$, then equation (11) is given by

$$v_t = b^1(v, v_1, \ldots, v_{n-1}, x, t, S, D(S), \ldots, D^{p-1}(S)).$$

Consider the *total derivative* operators

$$D_x = \frac{\partial}{\partial x} + \sum_{j \geqslant 0} u_{j+1} \frac{\partial}{\partial u_j}, \qquad D_t = \frac{\partial}{\partial t} + \sum_{j \geqslant 0} D_x^j (P(x, t, u, u_1, \ldots, u_p)) \frac{\partial}{\partial u_j}$$

and more general overdetermined systems

$$
\begin{aligned}
\frac{\partial w^i}{\partial x} &= a^i(w^1, \ldots, w^n, x, t, u), & i &= 1, \ldots, n, \\
\frac{\partial w^i}{\partial t} &= b^i(w^1, \ldots, w^n, x, t, u, \ldots, u_{p-1}), & i &= 1, \ldots, n,
\end{aligned}
\tag{16}
$$

consistent modulo (10). Clearly, an invertible change of variables

$$w^i \mapsto f^i(w^1, \ldots, w^n) \tag{17}$$

leads to a new system of form (16). Two systems related by such a change of variables are said to be *equivalent*.

System (16) is completely determined by the vector fields

$$A = \sum_{i=1}^n a^i(w^1, \ldots, w^n, x, t, u) \frac{\partial}{\partial w^i},$$

$$B = \sum_{i=1}^n b^i(w^1, \ldots, w^n, x, t, u, \ldots, u_{p-1}) \frac{\partial}{\partial w^i}.$$

Consistency of (16) modulo (10) is equivalent to the equation

$$[D_x + A, D_t + B] = 0. \tag{18}$$

Recall that two functions

$$M(x, t, u), \qquad N(x, t, u, \ldots, u_{p-1}) \tag{19}$$

with values in a Lie algebra $\mathfrak{g}$ constitute a *zero-curvature representation* (ZCR in short) for (10) if

$$[D_x + M, D_t + N] = D_x N - D_t M + [M, N] = 0. \tag{20}$$

Then each action $\rho \colon \mathfrak{g} \to D(W)$ and a choice of local coordinates $w^1, \ldots, w^n$ in W determine a consistent system of the form (16) with $A = \rho(M)$ and $B = \rho(N)$, since equation (18) follows from (20). Clearly, different choices of coordinates in W or isomorphic actions determine equivalent systems (16).

Definition 1. *Suppose that a ZCR (19) is given. For each $k \in \mathbb{N}$ we define a subalgebra $\mathfrak{g}^k$ of $\mathfrak{g}$ by induction on k as follows:*

- $\mathfrak{g}^0 = 0$,
- $\mathfrak{g}^1$ *is the subalgebra generated by all elements*

$$M(x, t, u) - M(x', t', u') \in \mathfrak{g},$$

  where x, t, u, x', t', u' run through all admissible (real or complex) values of the corresponding variables.
- $\mathfrak{g}^{k+1}$ *is generated by the subspaces $\mathfrak{g}^k$ and $[\mathfrak{g}^k, M(x, t, u)]$.*

Remark 1. Note that due to the definition of $\mathfrak{g}^1$ the space

$$\mathfrak{g}^k + [\mathfrak{g}^k, M(x, t, u)]$$

does not depend on the values of x, t, u.

Set also $\tilde{\mathfrak{g}} = \cup_{k \geqslant 0} \mathfrak{g}^k$.

Theorem 1. *Suppose that system (16) arises from an action $\rho \colon \mathfrak{g} \to D(W)$. Then the following two statements are equivalent.*

(1) There are $z = \left(w_0^1, \ldots, w_0^n\right) \in W$ and an invertible transformation (17) on a neighbourhood of z such that system (16) takes the form (14) with (15).

(2) There are $z \in W$ and a neighbourhood W_0 of z such that

$$\dim \operatorname{ev}_{\rho, z}(\tilde{\mathfrak{g}}) = n, \tag{21}$$

$$\forall a \in W_0 \qquad \dim \operatorname{ev}_{\rho, a}(\mathfrak{g}^{n-1}) < n, \tag{22}$$

$$\exists x_0, t_0, u_0 \qquad \frac{\partial}{\partial u}(\rho(M))(z, x_0, t_0, u_0) \neq 0. \tag{23}$$

In this case on a neighbourhood of z there is a function w such that

$$d_z w \neq 0, \qquad \rho(\mathfrak{g}^{n-1})(w) = 0, \tag{24}$$

and it is unique up to a change $w \mapsto g(w)$. The functions

$$\tilde{w}^i = \rho(M(x, t, u))^{i-1}(w), \qquad i = 1, \ldots, n, \tag{25}$$

do not depend on x, t, u and are local coordinates in which system (16) takes the desired form (14), (15).

Proof. If system (16) is of the form (14), (15) then by the definition of $\mathfrak{g}^k$ we obtain that on a neighbourhood of z the image of $\mathfrak{g}^k$ in each tangent space of W is spanned by $\partial/\partial w^{n-k+1}, \ldots, \partial/\partial w^n$. This obviously implies (21) and (22), and (23) follows from (15).

Conversely, let (21), (22) and (23) hold. Then existence of w and the fact that $\tilde{w}^1, \ldots, \tilde{w}^n$ are local coordinates follow from lemma 1 for $s = 1$ and $V = \rho(M(x, t, u))$. In particular, the functions $\tilde{w}^1, \ldots, \tilde{w}^{n-1}$ are invariant under $\rho(\mathfrak{g}^1)$. Combining this fact with the formula $\tilde{w}^{k+1} = \rho(M(x, t, u))(\tilde{w}^k)$, by induction on k one gets that each function $\tilde{w}^k$ does not depend on x, t, u.

It is easily seen that system (16) is of the form (14) in the coordinates $\tilde{w}^1, \ldots, \tilde{w}^n$. Finally condition (15) follows from (23). $\qquad\qquad\square$

Remark 2. In the above theorem, to construct a MT it is sufficient to know the restriction of ρ to some neighbourhood of z, and condition (21) implies that the action $\rho|_{\tilde{\mathfrak{g}}}$ is transitive on a neighbourhood of z.

Corollary 1. *If $\mathfrak{g}^m = \mathfrak{g}^{m+1}$ for some $m \geqslant 0$ (equivalently, $\tilde{\mathfrak{g}} = \mathfrak{g}^m$) then actions of $\mathfrak{g}$ cannot produce MTs of order greater than m.*

Proof. By theorem 1 and the above remark, a MT of order n is determined by a transitive action ρ of $\tilde{\mathfrak{g}}$ such that $\rho(\mathfrak{g}^{n-1})$ is not transitive. Since in our case $\tilde{\mathfrak{g}} = \mathfrak{g}^k$ for any $k \geqslant m$, there are no such actions for $n > m$. $\qquad\qquad\square$

Remark 3. If M does not depend on x, t then (23) follows from (21) and (22). Indeed, from the above proof in this case $\dim \operatorname{ev}_{\rho, z}(\mathfrak{g}^1) = 1$, which implies that

$$\exists u_0 \qquad \frac{\partial \rho(M)}{\partial u}(z, u_0) \neq 0.$$

4. Wahlquist–Estabrook algebras and MTs

4.1. General results

For a scalar translation-invariant equation

$$u_t = P(u, u_1, \ldots, u_p) \tag{26}$$

recall the definition of the Wahlquist–Estabrook algebra [18] from the point of view of [12]. Consider the equation

$$[D_x + A(u), D_t + B(u, u_1, \ldots, u_{p-1})] = 0, \tag{27}$$

where A, B are functions with values in a (not specified in advance) Lie algebra $\mathfrak{g}$. The algebra $\mathfrak{g}$ can also be the algebra of vector fields on a manifold W, then A and B are vector fields on W dependent on $u, \ldots, u_{p-1}$. Suppose that for any $\mathfrak{g}$ equation (27) implies

$$A = \sum_{i=1}^{k_1} f_i(u) F_i, \tag{28}$$

$$B = \sum_{i=1}^{k_2} g_i(u, u_1, \ldots, u_{p-1}) G_i, \tag{29}$$

where f_i, g_i are some fixed scalar functions, which do not depend on $\mathfrak{g}$, and F_i, G_i are elements of $\mathfrak{g}$. Moreover, suppose that functions (28) satisfy (27) if and only if some Lie algebra relations hold between the elements $F_1, \ldots, F_{k_1}, G_1, \ldots, G_{k_2}$. In this case, the quotient of the free Lie algebra generated by the letters F_i, G_j over these relations is called the Wahlquist–Estabrook algebra of (26) and is denoted by $\mathfrak{we}$. Functions (28), (29) constitute a ZCR with values in $\mathfrak{we}$ such that any consistent translation-invariant system (16) arises from this ZCR and some action of $\mathfrak{we}$.

The Wahlquist–Estabrook algebra exists for practically all known equations (see, e.g., [1, 12] and references therein).

By theorem 1 and remarks 2 and 3, we obtain the following result.

Theorem 2. *Translation-invariant MTs of order n of (26) are in one-to-one correspondence with actions ρ of $\mathfrak{we}$ such that $\rho(\tilde{\mathfrak{we}})$ is transitive and $\rho(\mathfrak{we}^{n-1})$ is not transitive. Locally isomorphic actions determine equivalent MTs.*

It turns out that non-translation-invariant MTs can also be described in terms of actions of $\mathfrak{we}$. By the construction of [10], with any system (14) we can associate the following translation-invariant system

$$\frac{\partial \hat{w}^1}{\partial x} = \frac{\partial \hat{w}^2}{\partial t} = 1, \qquad \frac{\partial \hat{w}^2}{\partial x} = \frac{\partial \hat{w}^1}{\partial t} = 0,$$

$$\frac{\partial w^i}{\partial x} = w^{i+1}, \qquad i = 1, \ldots, n-1,$$

$$\frac{\partial w^n}{\partial x} = a(w^1, \ldots, w^n, \hat{w}^1, \hat{w}^2, u),$$

$$\frac{\partial w^i}{\partial t} = b^i(w^1, \ldots, w^n, \hat{w}^1, \hat{w}^2, u, \ldots, u_{p-1}), \qquad i = 1, \ldots, n.$$

This system is consistent provided that the initial system (14) is consistent and equation (10) (equation (26)) is translation invariant. It is determined by an action ρ of $\mathfrak{we}$ on the manifold

with coordinates $\hat{w}^1, \hat{w}^2, w^1, \ldots, w^n$. If system (14) arises from a MT (12), (11) then we have (15) which, similar to the proof of theorem 1, implies

$$\dim \mathrm{ev}_{\rho,a}(\tilde{\mathfrak{w}}) = n, \qquad \dim \mathrm{ev}_{\rho,a}(\mathfrak{w}^{n-1}) = n - 1.$$

This observation implies the following (compare with corollary 1).

Theorem 3. *If for some n we have $\mathfrak{w}^n = \tilde{\mathfrak{w}}$ then any (not necessarily translation-invariant) MT over (26) is of order not greater than n.*

4.2. MTs of the KdV equation

Consider the KdV equation

$$u_t = u_3 + u_1 u$$

in the complex-analytic category. According to [17], we have

$$\mathfrak{w} = H \oplus \mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\lambda],$$

where H is the five-dimensional nilpotent Heisenberg algebra with the basis r_i, $i = -2, -1, 0, 1, 2$, and the commutator table

$$[r_{-1}, r_1] = [r_2, r_{-2}] = r_0, \qquad [r_i, r_j] = 0 \ \forall i + j \neq 0$$

and the Lie bracket in $\mathfrak{sl}_2 \otimes \mathbb{C}[\lambda]$ is defined as follows:

$$[g_1 \otimes f_1(\lambda), g_2 \otimes f_2(\lambda)] = [g_1, g_2] \otimes f_1(\lambda) f_2(\lambda), \qquad g_i \in \mathfrak{sl}_2, \qquad f_i(\lambda) \in \mathbb{C}[\lambda].$$

Below an element $g \otimes f(\lambda)$ of $\mathfrak{sl}_2 \otimes \mathbb{C}[\lambda]$ will be written simply as $gf(\lambda)$.

The universal ZCR reads

$$A(u) = X_1 + \tfrac{1}{3}u X_2 + \tfrac{1}{6}u^2 X_3, \tag{30}$$

$$X_1 = r_1 - \tfrac{1}{2}y + \tfrac{1}{2}z\lambda, \qquad X_2 = r_{-1} + z, \qquad X_3 = r_{-2}, \tag{31}$$

where h, y, z is a basis of $\mathfrak{sl}_2$ with the relations $[h, y] = 2y$, $[h, z] = -2z$, $[y, z] = h$. Here the form of $B(u, u_1, u_2)$ in (29) is not important for us.

From (30) and (31), using definition 3, one obtains

$$\mathfrak{w}^1 = \langle X_2, X_3 \rangle, \qquad \mathfrak{w}^2 = \langle r_{-2}, r_{-1}, z, 2r_0 + h \rangle,$$

$$\mathfrak{w}^3 = \mathfrak{w}^k = \tilde{\mathfrak{w}} = \langle \mathfrak{sl}_2 \otimes \mathbb{C}[\lambda], r_{-2}, r_{-1}, r_0 \rangle \qquad \forall k \geqslant 3.$$

By theorem 3, any MT of the KdV equation is of order not greater than 3. For translation-invariant MTs this was proved in [9].

Let us explain how our method of constructing MTs includes that of [2]. Set $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\lambda]$. We have the natural projection $\mathfrak{w} \to \mathfrak{g}$ that maps H to zero. Combining it with the above ZCR, we obtain a ZCR with values in $\mathfrak{g}$ whose x-part reads

$$M(u) = -\tfrac{1}{2}y + \tfrac{1}{2}z\lambda + \tfrac{1}{3}uz.$$

For this ZCR we have

$$\mathfrak{g}^1 = \langle z \rangle, \qquad \mathfrak{g}^2 = \langle z, h \rangle, \qquad \mathfrak{g}^3 = \tilde{\mathfrak{g}} = \mathfrak{g}.$$

By theorem 2, each transitive action ρ of $\mathfrak{g} = \tilde{\mathfrak{g}}$ on a manifold of dimension $n \leqslant 3$ determines a MT for the KdV equation, because the algebra $\rho(\mathfrak{g}^{n-1})$ is of dimension $\leqslant n - 1$ and cannot be transitive.

According to [6], for a transitive action $\rho: \mathfrak{g} \to D(W)$ the image $\rho(\mathfrak{g})$ is finite dimensional and is of the form

$$\mathfrak{sl}_2 \otimes \mathbb{C}[\lambda]/(f(\lambda)), \qquad f(\lambda) \in \mathbb{C}[\lambda], \tag{32}$$

where $(f(\lambda))$ is the ideal of $\mathbb{C}[\lambda]$ generated by $f(\lambda)$. Let

$$f(\lambda) = a \prod_{s=1}^{k} (\lambda - e_s)^{k_s}, \quad a, e_s \in \mathbb{C}, \quad a \neq 0, \quad e_i \neq e_j \forall i \neq j.$$

Then Lie algebra (32) is isomorphic to

$$\bigoplus_{s=1}^{k} \mathfrak{sl}_2 \otimes \mathbb{C}[\lambda]/(\lambda^{k_s}). \tag{33}$$

The Lie groups $\prod_s \mathrm{SL}_2(\mathbb{C}[\lambda]/(\lambda^{k_s}))$ that appear in [2] have (33) as their Lie algebras. Thus, construction of MTs arising from this ZCR is reduced to local description of homogeneous spaces of dim $\leqslant 3$ of these Lie groups. This description and the corresponding MTs are presented in [2].

A translation-invariant MT over the KdV equation belongs to the list of MTs in [2] if and only if for the corresponding action ρ of $\mathfrak{we}$ we have $\rho(H) = 0$.

Example 1. Consider the following action of $\mathfrak{we}$ on $\mathbb{C}$

$$\mathfrak{g} \to 0, \qquad r_i \mapsto 0, \qquad i = -2, 0, 1, 2, \qquad r_{-1} \mapsto \frac{\partial}{\partial w},$$

where w is a coordinate in $\mathbb{C}$. The corresponding MT is

$$u = 3v_1, \qquad v_t = v_3 + \tfrac{3}{2} v_1^2$$

and does not belong to the list of MTs in [2].

4.3. MTs of a Harry Dym type equation

Consider the equation [4, 11]

$$u_t = u^{5/2} u_5 \tag{34}$$

in the complex-analytic category. There are Bäcklund transformations connecting (34) with the Sawada–Kotera and Kaup–Kupershmidt equations [4].

According to [4], the Wahlquist–Estabrook algebra of (34) is the direct sum $\mathbb{C}^2 \oplus \mathfrak{g}$, where $\mathbb{C}^2$ is a commutative algebra with a basis C_1, C_2 and $\mathfrak{g}$ is the 'positive part' of the twisted affine algebra $A_2^{(2)}$. In other words, the algebra $\mathfrak{g}$ is isomorphic to a subalgebra of $\mathfrak{sl}_3(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\lambda]$ generated by the two elements

$$X_1 = \begin{Vmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \qquad X_2 = \begin{Vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{Vmatrix} \tag{35}$$

satisfying the relations

$$(\mathrm{ad}X_1)^2 X_2 = 0, \tag{36}$$

$$(\mathrm{ad}X_2)^5 X_1 = 0. \tag{37}$$

The corresponding ZCR reads

$$A(u) = C_1 u^{-\frac{1}{2}} + X_1 u^{-\frac{3}{2}} + X_2,$$

the form of $B(u, u_1, u_2, u_3, u_4)$ can be found in [4] and is not important for our purposes.

The subalgebra $\mathfrak{we}^k$, $k \geqslant 1$, is generated by the elements

$$C_1, \qquad (\mathrm{ad}X_2)^i X_1, \qquad i = 0, \dots, k - 1.$$

Combining this with relation (37), we obtain $\mathfrak{w}\mathfrak{e}^5 = \mathfrak{w}\mathfrak{e}^6 = \tilde{\mathfrak{w}}\mathfrak{e}$. By theorem 3, we obtain that any MT over (34) is of order not greater than 5.

Let us construct a MT over (34). Consider the homomorphism $\mathfrak{w}\mathfrak{e} \to \mathfrak{sl}_3$ that maps C_i to zero and substitutes $\lambda = 1$ in (35). Combining the standard action of $\mathfrak{sl}_3$ on $\mathbb{C}^3$ with this homomorphism, we obtain the following transitive action:

$$\rho: \mathfrak{w}\mathfrak{e} \to D(\mathbb{C}^3), \qquad \rho(C_i) = 0,$$

$$\rho(X_1) = w^2 \frac{\partial}{\partial w^1}, \qquad \rho(X_2) = w^3 \frac{\partial}{\partial w^2} + w^1 \frac{\partial}{\partial w^3}.$$

According to theorem 1, to get a MT from this action we need to find a non-constant function w on $\mathbb{C}^3$ such that $\rho(\mathfrak{w}\mathfrak{e}^2)(w) = 0$. Since the algebra $\rho(\mathfrak{w}\mathfrak{e}^2)$ is commutative and spanned by the vector fields

$$\rho(X_1) = w^2 \frac{\partial}{\partial w^1}, \qquad \rho([X_2, X_1]) = w^3 \frac{\partial}{\partial w^1} - w^2 \frac{\partial}{\partial w^3},$$

we can take $w = w^2$. By formula (25) for $M(x, t, u) = A(u)$, we have $\tilde{w}^1 = w^2$, $\tilde{w}^2 = w^3$, $\tilde{w}^3 = w^1$. Rewriting the vector field $\rho(A(u))$ in these coordinates, we obtain that the x-part of the corresponding system (14) is

$$\frac{\partial \tilde{w}^1}{\partial x} = \tilde{w}^2, \qquad \frac{\partial \tilde{w}^2}{\partial x} = \tilde{w}^3, \qquad \frac{\partial \tilde{w}^3}{\partial x} = u^{-\frac{3}{2}} \tilde{w}^1. \tag{38}$$

Applying the substitution (13) for $w^i = \tilde{w}^i$, from (38) we obtain the MT

$$u = \left(\frac{v}{v_3} \right)^{\frac{2}{3}}.$$

The corresponding equation (11) can be obtained either by straightforward computation or using the vector field $\rho(B(u, u_1, u_2, u_3, u_4))$ as described in section 3. The answer is

$$v_t = -9v D^2 \left(\left(\frac{v}{v_3} \right)^{\frac{2}{3}} \right) + \frac{9}{2} v_1 D \left(\left(\frac{v}{v_3} \right)^{\frac{2}{3}} \right) - \frac{3}{2} v_2 \left(\frac{v}{v_3} \right)^{\frac{2}{3}}, \tag{39}$$

where $D = \sum_i v_{i+1} \partial / \partial v_i$.

5. MTs of non-scalar evolution systems

5.1. MTs from ZCRs

A MT (3) of non-scalar systems cannot always be written in some simple analogue of the form (14), and, therefore, not all MTs of non-scalar systems can be described by our method. In this section, we study the MTs that can be written in a form analogous to (14).

Consider an evolution system (2) and a system

$$\frac{\partial w^j}{\partial x} = w^{i+1}, \qquad j = 1, \ldots, n - s, \tag{40}$$

$$\frac{\partial w^{n-s+i}}{\partial x} = A^i(w^1, \ldots, w^n, x, t, u^1, \ldots, u^k), \qquad i = 1, \ldots, s, \tag{41}$$

$$\frac{\partial w^l}{\partial t} = b^i(w^1, \ldots, w^n, x, t, u^1, \ldots, u^k, u_1^1, \ldots, u_1^k, \cdots), \qquad l = 1, \ldots, n. \tag{42}$$

consistent modulo (2). By analogy with (15), suppose that the following condition holds:

$$\exists w_0^1, \ldots, w_0^n, x_0, t_0, u_0^1, \ldots, u_0^k$$

$$\mathrm{rank} \left\| \frac{\partial A^i}{\partial u^j} \right\| (w_0^1, \ldots, w_0^n, x_0, t_0, u_0^1, \ldots, u_0^k) = s. \tag{43}$$

In particular, from (43) we have $s \leqslant k$.

Then one gets a MT over (2) as follows. Taking into account (43), by the implicit function theorem, there are $1 \leqslant j_1 < \cdots < j_s \leqslant k$ such that $u^{j_1}, \ldots, u^{j_s}$ can locally be expressed from (41) in terms of

$$x, \quad t, \quad w^1, \ldots, w^n, \quad \frac{\partial w^{n-s+i}}{\partial x}, \quad i = 1, \ldots, s.$$

Let $n_1 < \cdots < n_{k-s}$ be such that $\{j_1, \ldots, j_s, n_1, \ldots, n_{k-s}\} = \{1, \ldots, k\}$. Substitute everywhere

$$w^j = v_{j-1}^1, \qquad \frac{\partial w^{n-s+1}}{\partial x} = v_{n-s}^1, \qquad j = 1, \ldots, n-s,$$

$$w^{n-s+j} = v^j, \qquad \frac{\partial w^{n-s+j}}{\partial x} = v_1^j, \qquad j = 2, \ldots, s,$$

$$u^{n_j} = v^{s+j}, \qquad j = 1, \ldots, k-s.$$

Thus we expressed u^i in terms of x, t, v_l^j, that is, we got (3). Finally, system (1) is obtained from (42).

The total derivative operators are now

$$D_x = \frac{\partial}{\partial x} + \sum_{\substack{i=1,\ldots,k, \\ j \geqslant 0}} u_{j+1}^i \frac{\partial}{\partial u_j^i}, \qquad D_t = \frac{\partial}{\partial t} + \sum_{\substack{i=1,\ldots,k, \\ j \geqslant 0}} D_x^j (P^i(x, t, u, u_1, \ldots, u_p)) \frac{\partial}{\partial u_j^i}.$$

Similarly to (16) we consider systems

$$\frac{\partial w^i}{\partial x} = a^i(w^1, \ldots, w^n, x, t, u^1, \ldots, u^k), \qquad i = 1, \ldots, n,$$

$$\frac{\partial w^i}{\partial t} = b^i(w^1, \ldots, w^n, x, t, u^1, \ldots, u^k, u_1^1, \ldots, u_1^k, \ldots), \qquad i = 1, \ldots, n, \tag{44}$$

consistent modulo (2) with the same equivalence relation (17).

Two functions

$$M(x, t, u^1, \ldots, u^k), \qquad N(x, t, u^1, \ldots, u^k, u_1^1, \ldots, u_1^k, \ldots)$$

with values in a Lie algebra $\mathfrak{g}$ constitute a ZCR for (2) if equation (20) holds. Each action $\rho : \mathfrak{g} \to D(W)$ and a choice of local coordinates $w^1, \ldots, w^n$ in W determine a consistent system of the form (44) as follows:

$$\rho(M) = \sum_{i=1}^n a^i(w^1, \ldots, w^n, x, t, u^1, \ldots, u^k) \frac{\partial}{\partial w^i},$$

$$\rho(N) = \sum_{i=1}^n b^i(w^1, \ldots, w^n, x, t, u^1, \ldots, u^k, u_1^1, \ldots, u_1^k, \ldots) \frac{\partial}{\partial w^i}.$$

Similar to definition 1, we define subalgebras $\mathfrak{g}^k$ of $\mathfrak{g}$ by induction on k as follows:

- $\mathfrak{g}^0 = 0$,
- $\mathfrak{g}^1$ is the subalgebra generated by all elements

$$M(x, t, u^1, \ldots, u^k) - M(x', t', (u^1)', \ldots, (u^k)') \in \mathfrak{g},$$

- $\mathfrak{g}^{k+1}$ is generated by the subspaces $\mathfrak{g}^k$ and

$$[\mathfrak{g}^k, M(x, t, u^1, \ldots, u^k)].$$

Set also $\tilde{\mathfrak{g}} = \cup_{k \geqslant 0} \mathfrak{g}^k$.

Theorem 4. *Suppose that system (44) arises from an action $\rho \colon \mathfrak{g} \to D(W)$ and there are $x_0, t_0, u_0^1, \ldots, u_0^k, z \in W$ and a neighbourhood W_0 of z such that*

$$\dim \mathrm{ev}_{\rho,z}(\tilde{\mathfrak{g}}) = n, \qquad \forall a \in W_0 \quad \dim \mathrm{ev}_{\rho,a}(\mathfrak{g}^{n-s}) < n,$$

where s is the dimension of the subspace

$$\left\langle \frac{\partial \rho(M)}{\partial u^i}(z, x_0, t_0, u_0^1, \ldots, u_0^k) \middle| i = 1, \ldots, k \right\rangle \subset T_z W. \tag{45}$$

Then on a neighbourhood of z there is a function w such that

$$d_z w \neq 0, \qquad \rho(\mathfrak{g}^{n-s})(w) = 0,$$

and it is unique up to a change $w \mapsto g(w)$. The functions

$$\tilde{w}^i = \rho(M(x, t, u^1, \ldots, u^k))^{i-1}(w), \qquad i = 1, \ldots, n - s + 1,$$

do not depend on x, t, u^i. One can find functions $\tilde{w}^{n-s+2}, \ldots, \tilde{w}^n$ such that $\tilde{w}^1, \ldots, \tilde{w}^n$ form a system of coordinates on a neighbourhood of z. The initial system (16) takes the form (40), (41), (42), (43) in these coordinates.

Proof. Similar to theorem 1, this follows from lemma 1 for $V = \rho(M(x, t, u^1, \ldots, u^k))$. Condition $m_1 = \dim \mathrm{ev}_{\rho,z}(\mathfrak{g}^1) \geqslant s$ holds because the space $\mathrm{ev}_{\rho,z}(\mathfrak{g}^1)$ includes subspace (45). $\qquad \square$

5.2. MTs of the Kersten–Krasilshchik system

The following system was introduced in [8] and proved to be integrable

$$
\begin{aligned}
u_t^1 &= -u_3^1 + 6u^1 u_1^1 - 3u^2 u_3^2 - 3u_1^2 u_2^2 + 3u_1^1 (u^2)^2 + 6u^1 u^2 u_1^2, \\
u_t^2 &= -u_3^2 + 3(u^2)^2 u_1^2 + 3u^1 u_1^2 + 3u_1^1 u^2.
\end{aligned}
\tag{46}
$$

Denote by E_{ij}, $i, j = 1, 2, 3$, the (3×3)-matrix with (i, j)-entry equal to 1 and other entries equal to 0. Let $\mathfrak{g}$ be the five-dimensional Lie subalgebra of $\mathfrak{sl}_3(\mathbb{C})$ spanned by the elements

$$e = E_{12}, \qquad n_1 = E_{13}, \qquad f = E_{21}, \qquad n_2 = E_{23}, \qquad h = E_{11} - E_{22}.$$

An $\mathfrak{sl}_3$-valued ZCR dependent on a parameter λ was constructed for (46) in [7]. For $\lambda = 0$ one obtains the following ZCR:

$$M = ((u^2)^2 - u^1)e - f - u^2 n_1$$

and

$$
\begin{aligned}
N = {} & \left(u_2^1 + u^2 u_2^2 + \left(u_1^2\right)^2 - 2(u^1)^2 + (u^2)^4 + (u^2)^2 u^1\right)e \\
& - \left(u_1^1 + u^2 u_1^2\right)h - ((u^2)^2 + 2u^1)f + \left(u_2^2 - (u^2)^3 - 2u^1 u^2\right)n_1 + u_1^2 n_2.
\end{aligned}
$$

Let us construct MTs for (46) from this ZCR using theorem 4. We have

$$\mathfrak{g}_1 = \langle e, n_1 \rangle, \qquad \mathfrak{g}_2 = \langle h, e, n_1, n_2 \rangle, \qquad \mathfrak{g}_3 = \tilde{\mathfrak{g}} = \mathfrak{g}. \tag{47}$$

Every subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of codimension 3 such that

$$\mathfrak{g}^1 \cap \mathfrak{h} = 0 \tag{48}$$

determines a MT as follows. Consider Lie groups $H \subset G$ corresponding to the Lie algebras $\mathfrak{h} \subset \mathfrak{g}$ and set $W = G/H, z = H \in G/H$. We have the natural transitive action of $\rho: \mathfrak{g} \to D(W)$. From (47) and (48) we obtain that the dimension of space (45) is equal to 2. Since $\dim \mathfrak{g}^1 = 2$ and $\dim W = 3$, the algebra $\rho(\mathfrak{g}^1)$ is not transitive on W. Therefore, all conditions of theorem 4 are satisfied, and ρ determines a MT with $n = 3, s = 2$.

For example, set $\mathfrak{h} = \langle n_2, f - n_1 \rangle$. The corresponding action in local coordinates reads

$$e \mapsto \frac{\partial}{\partial w^2}, \qquad n_1 \mapsto \frac{\partial}{\partial w^3}, \qquad n_2 \mapsto w^2 \frac{\partial}{\partial w^3},$$

$$f \mapsto w^2 \frac{\partial}{\partial w^1} - (w^2)^2 \frac{\partial}{\partial w^2} + \left(e^{-3w^1} - w^2 w^3 \right) \frac{\partial}{\partial w^3},$$

$$h \mapsto \frac{\partial}{\partial w^1} - 2w^2 \frac{\partial}{\partial w^2} - w^3 \frac{\partial}{\partial w^3}.$$

Indeed, the isotropy subalgebra of the point $z = (0, 0, 0)$ is equal to $\mathfrak{h}$. The MT reads

$$u^1 = v_2^1 + \left(v_1^1 \right)^2 + \left(v_1^2 + v_1^1 v^2 - e^{-3v^1} \right)^2, \qquad u^2 = v_1^2 + v_1^1 v^2 - e^{-3v^1}.$$

The corresponding system (1) is cumbersome and can be obtained from $\rho(N)$ as described in subsection 5.1.

Acknowledgment

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